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# A complement to monotonicity of generalized Furuta-type operator functions (Application of Geometry to Operator Theory)

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# A complement to monotonicity of generalized Furuta-type operator functions

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## Abstract

Recently, Furuta obtained the results on monotonicity of a generalized Furuta-type operator function  $F(\lambda, \mu) = A^{-\lambda} \sharp_{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} (A^{\frac{-t}{2}} B^p A^{\frac{t}{2}})^{\mu}$ .

In this report, we shall show the result which considers a domain not considered in Furuta's one as follows: Let  $A \geq B \geq 0$  with  $A > 0$ ,  $t \in [0, 1]$  and  $p \geq 1$ . Then  $F(\lambda, \mu)$  satisfies

$$F(q, w) \geq F(t, 1) \geq F(r, s) \geq F(r', s')$$

for any  $s' \geq s \geq 1$ ,  $r' \geq r \geq t$ ,  $\frac{1-t}{p-t} \leq w \leq 1$  and  $t-1 \leq q \leq t$ .

We shall also discuss an equivalence relation related to Ando-Hiai inequality.

## 1 Introduction

This report is based on our recent paper [21] and preprint [4].

In this report, a capital letter means a bounded linear operator on a complex Hilbert space  $\mathcal{H}$ . An operator  $T$  is said to be positive (denoted by  $T \geq 0$ ) if  $(Tx, x) \geq 0$  for all  $x \in \mathcal{H}$ , and also an operator  $T$  is said to be strictly positive (denoted by  $T > 0$ ) if  $T$  is positive and invertible.

The following Löwner-Heinz theorem is a famous order preserving operator inequality.

$$A \geq B \geq 0 \text{ implies } A^{\alpha} \geq B^{\alpha} \text{ for any } \alpha \in [0, 1].$$

In 1987, Furuta inequality [11] is established as an extension of Löwner-Heinz theorem.

**Theorem 1.A** (Furuta inequality [11]).

If  $A \geq B \geq 0$ , then for each  $r \geq 0$ ,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for  $p \geq 0$  and  $q \geq 1$  with  $(1+r)q \geq p+r$ .

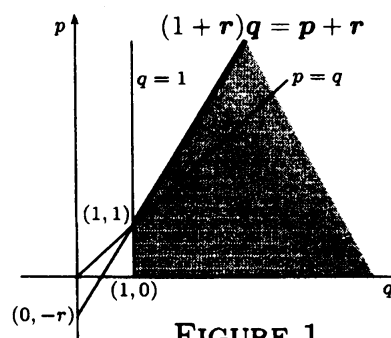


FIGURE 1

By putting  $r = 0$  in Theorem 1.A, we can get Löwner-Heinz theorem. Alternative proofs of Theorem 1.A are given in [2, 22] and also an elementary one page proof in [12]. Tanahashi [25] showed that the domain drawn for  $p, q$  and  $r$  in the Figure 1 is the best possible one for Theorem 1.A.

As stated in [22], when  $A > 0$  and  $B \geq 0$ , (ii) of Theorem 1.A can be arranged in terms of  $\alpha$ -power mean  $\sharp_\alpha$  for  $\alpha \in [0, 1]$  introduced by Kubo-Ando [24] as  $A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$  :

$$A \geq B \geq 0 \text{ with } A > 0 \text{ implies } A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A \text{ for } p \geq 1 \text{ and } r \geq 0. \quad (\text{F})$$

Next we shall discuss weaker order than usual one  $A \geq B$ . For  $A, B > 0$ , the order  $\log A \geq \log B$  is called chaotic order. It is well known that chaotic order is weaker than usual one since  $\log t$  is an operator monotone function for  $t > 0$ .

As a characterization of chaotic order, in [3] and [13] (see also [5, 27]), they showed the following: For  $A, B > 0$ ,

$$\log A \geq \log B \text{ if and only if } A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I \text{ for all } p \geq 0 \text{ and } r \geq 0, \quad (1.1)$$

and also

$$\log A \geq \log B \text{ implies } A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p \leq B^\delta \text{ for } p \geq \delta \geq 0 \text{ and } r \geq 0.$$

We remark that an excellent proof of (1.1) which used only Theorem 1.A was shown in [27]. We can summarize above results as follows: For  $A, B > 0$ ,

$$\begin{aligned} A \geq B & \implies A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A \text{ for } p \geq 1 \text{ and } r \geq 0. \\ \Downarrow \\ A^q \geq B^q \ (q \in (0, 1)) & \implies A^{-r} \sharp_{\frac{q+r}{p+r}} B^p \leq B^q \leq A^q \text{ for } p \geq q \text{ and } r \geq 0. \\ \Downarrow \\ \log A \geq \log B & \iff (1.1): A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I \text{ for all } p \geq 0 \text{ and } r \geq 0. \\ \Downarrow \\ A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p \leq B^\delta & \\ \text{for } p \geq \delta \geq 0 \text{ and } r \geq 0. & \end{aligned}$$

## 2 Equivalence relation related to Ando-Hiai inequality

In 1994, Ando and Hiai [1] have shown the following inequality.

**Theorem 2.A** (Ando-Hiai inequality [1]). For  $A, B > 0$ ,

$$A \sharp_{\alpha} B \leq I \text{ for } \alpha \in (0, 1) \text{ implies } A^r \sharp_{\alpha} B^r \leq I \text{ for } r \geq 1. \quad (\text{AH})$$

By (AH), they obtained that for  $A, B > 0$ ,

$$A^{-1} \sharp_{\frac{1}{p}} A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}} \leq I \text{ implies } A^{-r} \sharp_{\frac{1}{p}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^r \leq I \text{ for } p \geq 1 \text{ and } r \geq 1, \quad (\text{AH}')$$

that is,

$$A \geq B > 0 \text{ implies } A^r \geq \{A^{\frac{r}{2}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^r A^{\frac{r}{2}}\}^{\frac{1}{p}} \text{ for } p \geq 1 \text{ and } r \geq 1. \quad (\text{AH}'')$$

We remark that (AH'') is equivalent to the main result of log majorization.

In [8], it was pointed out that the following (C) is the essence of (F).

$$A \geq B > 0 \text{ implies } A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I \text{ for } p \geq 0 \text{ and } r \geq 0. \quad (\text{C})$$

We remark that (F) implies (C) immediately by Löwner-Heinz theorem. It was shown in [7] that an equivalence relation holds between (AH) and (F) via (C). Here we can obtain an equivalence relation between (AH) and (C) without using (F).

**Theorem 2.1** ([4]). (AH) is equivalent to (C).

*Proof of Theorem 2.1.* Suppose that (C) holds and that  $A \sharp_{\alpha} B \leq I$ . We put  $p = \frac{1}{\alpha} > 1$ . Then the assumption  $A \sharp_{\alpha} B \leq I$  says that

$$B_1 = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} \leq A^{-1} = A_1.$$

Applying (C) to  $A_1 \geq B_1$ , we have

$$A_1^{-r} \sharp_{\frac{r}{p+r}} B_1^p \leq I \text{ for } r \geq 0.$$

Moreover it follows that for  $p \geq 1$  and  $r \geq 0$ ,

$$\begin{aligned} A_1^{-r} \sharp_{\frac{1+r}{p+r}} B_1^p &= B_1^p \sharp_{\frac{p-1}{p+r}} A_1^{-r} = B_1^p \sharp_{\frac{p-1}{p}} (B_1^p \sharp_{\frac{p}{p+r}} A_1^{-r}) \\ &= B_1^p \sharp_{\frac{p-1}{p}} (A_1^{-r} \sharp_{\frac{r}{p+r}} B_1^p) \leq B_1^p \sharp_{\frac{p-1}{p}} I = B_1 \leq A_1. \end{aligned}$$

Summing up the above discussion, for each  $p > 1$ ,

$$A \sharp_{\frac{1}{p}} B \leq I \text{ implies } A^r \sharp_{\frac{1+r}{p+r}} A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq A^{-1}, \text{ or } A^{r+1} \sharp_{\frac{1+r}{p+r}} B \leq I \text{ for } r \geq 0.$$

Noting that

$$B \sharp_{\frac{p-1}{p+r}} A^{r+1} = A^{r+1} \sharp_{\frac{1+r}{p+r}} B \leq I,$$

we apply it for  $p_1 = \frac{p+r}{p-1}$  in the following way;

$$I \geq B^{r+1} \sharp_{\frac{1+r}{p_1+r}} A^{r+1} = A^{r+1} \sharp_{\frac{1}{p}} B^{r+1}$$

by  $1 - \frac{1+r}{p_1+r} = \frac{1}{p}$ . Namely we obtain (AH).

(AH)  $\Rightarrow$  (C) has been already shown in [7]. But we cite it for the sake of convenience: It suffices to show that (C) holds for  $p, r > 1$  under the assumption  $A \geq B > 0$  because it holds for  $0 \leq p, r \leq 1$  by Löwner-Heinz theorem. So we take arbitrary  $p, r > 1$ , and put  $\alpha = \frac{r}{p+r}$  and  $q = \max\{p, r\}$ . Then, as noted in above, if  $A \geq B > 0$ , then (C) holds for  $p_1 = \frac{p}{q}$  and  $r_1 = \frac{r}{q}$ , i.e.,

$$A^{-r_1} \sharp_{\frac{r_1}{p_1+r_1}} B^{p_1} \leq I.$$

We here apply (AH) to this, that is, we have

$$I \geq A^{-r_1 q} \sharp_{\frac{r_1 q}{p_1 q + r_1 q}} B^{p_1 q} = A^{-r} \sharp_{\frac{r}{p+r}} B^p,$$

as desired. □

### 3 A complement to monotonicity of generalized Furuta-type operator functions

In 1995, Furuta [14] obtained the following theorem.

**Theorem 3.A** (Grand Furuta inequality [14]). *If  $A \geq B \geq 0$  with  $A > 0$ , then for each  $t \in [0, 1]$  and  $p \geq 1$ ,*

$$F(r, s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}} \quad (3.1)$$

*is decreasing for  $r \geq t$  and  $s \geq 1$ , and*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \quad (3.2)$$

*holds for  $r \geq t$  and  $s \geq 1$ .*

Theorem 3.A is established as a generalization of both Furuta inequality (F) and Ando-Hiai inequality (AH''). In fact, Theorem 3.A leads (F) by putting  $t = 0$  and  $s = 1$ , and also leads (AH'') by putting  $t = 1$  and  $s = r$ . An alternative proof of Theorem 3.A is given in [6] and an elementary one-page proof of (3.2) is in [15]. Related results to Theorem 3.A are shown in [16, 18, 19, 20, 29] and so on. It is shown in [26] (see also [10, 28]) that the outside exponents of (3.2) are the best possible. We remark that (3.1) can be rewritten by using  $\alpha$ -power mean as follows:

$$F(\lambda, \mu) = A^{-\lambda} \sharp_{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\mu}. \quad (3.1')$$

Related to Theorem 3.A, the following result was shown in [23, 9].

**Theorem 3.B** ([23, 9]). Let  $A \geq B \geq 0$  with  $A > 0$ ,  $t \in [0, 1]$  and  $p \geq 1$ . Then

$$A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^t \sharp_{\frac{1-t}{p-t}} B^p$$

for  $s \geq 1$  and  $r \geq t$ , where  $A \natural_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}}$  for  $s \in \mathbb{R}$ .

Very recently, as a generalization of Theorem 3.B, the following theorem was shown on monotonicity of a generalized Furuta-type operator function (3.1').

**Theorem 3.C** ([17]). Define  $F(\lambda, \mu)$  as (3.1'). Let  $A \geq B \geq 0$  with  $A > 0$ ,  $t \in [0, 1]$  and  $p \geq 1$ . Then  $F(\lambda, \mu)$  satisfies the following properties:

$$(i) \quad F(r, w) \geq F(r, 1) \geq F(r, s) \geq F(r, s')$$

holds for any  $s' \geq s \geq 1$ ,  $r \geq t$  and  $\frac{1-t}{p-t} \leq w \leq 1$ .

$$(ii) \quad F(q, s) \geq F(t, s) \geq F(r, s) \geq F(r', s)$$

holds for any  $r' \geq r \geq t$ ,  $s \geq 1$  and  $t-1 \leq q \leq t$ .

$F(\lambda, \mu)$  is not always decreasing for  $\frac{1-t}{p-t} \leq \lambda \leq 1$  and  $t-1 \leq \mu \leq t$  (see [17]). But Theorem 3.C says that we can compare  $F(r, w)$  with  $F(r, 1)$  for  $\frac{1-t}{p-t} \leq w \leq 1$ , and  $F(q, s)$  with  $F(t, s)$  for  $t-1 \leq q \leq t$ . We remark that Theorem 3.C leads Theorem 3.B by putting  $w = \frac{1-t}{p-t}$  in (i) or  $q = 0$  in (ii).

Here, we shall consider a domain not considered in Theorem 3.C, that is, we shall show that we can also compare  $F(q, w)$  with  $F(t, 1)$  for  $\frac{1-t}{p-t} \leq w \leq 1$  and  $t-1 \leq q \leq t$ .

**Theorem 3.1** ([21]). Define  $F(\lambda, \mu)$  as (3.1'). Let  $A \geq B \geq 0$  with  $A > 0$ ,  $t \in [0, 1]$  and  $p \geq 1$ . Then  $F(\lambda, \mu)$  satisfies

$$F(q, w) \geq F(t, 1) \geq F(r, s) \geq F(r', s')$$

for any  $s' \geq s \geq 1$ ,  $r' \geq r \geq t$ ,  $\frac{1-t}{p-t} \leq w \leq 1$  and  $t-1 \leq q \leq t$ .

*Proof of Theorem 3.1.* We have only to show  $F(q, w) \geq F(t, 1)$  since  $F(t, 1) \geq F(r, s) \geq F(r', s')$  is just Theorem 3.A.

By Löwner-Heinz theorem,  $A^{t-q} \geq B^{t-q}$  since  $t-q \in [0, 1]$  and  $A^t \geq B^t$  since  $t \in [0, 1]$ , so that we have

$$\begin{aligned} F(q, w) &= A^{-q} \sharp_{\frac{1-t+q}{(p-t)w+q}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^w = A^{\frac{-t}{2}} \{A^{t-q} \sharp_{\frac{1-t+q}{(p-t)w+q}} (A^t \sharp_w B^p)\} A^{\frac{-t}{2}} \\ &\geq A^{\frac{-t}{2}} \{B^{t-q} \sharp_{\frac{1-t+q}{(p-t)w+q}} (B^t \sharp_w B^p)\} A^{\frac{-t}{2}} = A^{\frac{-t}{2}} B A^{\frac{-t}{2}} = A^{-t} \sharp_{\frac{1}{p}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) \\ &= F(t, 1). \end{aligned}$$

Hence the proof is complete.  $\square$

Figure 2 expresses the domain of  $\lambda$  and  $\mu$  in which Theorem 3.A, Theorem 3.C and Theorem 3.1 hold.

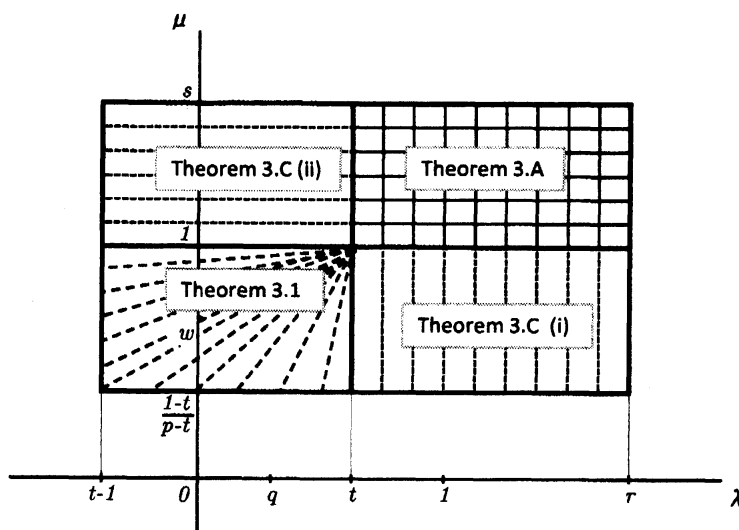


FIGURE 2

## References

- [1] T. Ando and F. Hiai, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., **197**, **198** (1994), 113–131.
- [2] M. Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory, **23** (1990), 67–72.
- [3] M. Fujii, T. Furuta and E. Kamei, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl., **179** (1993), 161–169.
- [4] M. Fujii, M. Ito, E. Kamei and A. Matsumoto, *Operator inequalities related to Ando-Hiai inequality*, preprint.
- [5] M. Fujii, J. F. Jiang and E. Kamei, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc., **125** (1997), 3655–3658.
- [6] M. Fujii and E. Kamei, *Mean theoretic approach to the grand Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 2751–2756.

- [7] M. Fujii and E. Kamei, *Ando-Hiai inequality and Furuta inequality*, Linear Algebra Appl., **416** (2006), 541–545.
- [8] M. Fujii, E. Kamei and R. Nakamoto, *An analysis on the internal structure of the celebrated Furuta inequality via operator mean*, Sci. Math. Jpn., **62** (2005), 421–427.
- [9] M. Fujii, E. Kamei and R. Nakamoto, *Grand Furuta inequality and its variant*, J. Math. Inequal., **1** (2007), 437–441.
- [10] M. Fujii, A. Matsumoto and R. Nakamoto, *A short proof of the best possibility for the grand Furuta inequality*, J. Inequal. Appl., **4** (1999), 339–344.
- [11] T. Furuta,  *$A \geq B \geq 0$  assures  $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$  for  $r \geq 0$ ,  $p \geq 0$ ,  $q \geq 1$  with  $(1 + 2r)q \geq p + 2r$* , Proc. Amer. Math. Soc., **101** (1987), 85–88.
- [12] T. Furuta, *An elementary proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci., **65** (1989), 126.
- [13] T. Furuta, *Applications of order preserving operator inequalities*, Oper. Theory Adv. Appl., **59** (1992), 180–190.
- [14] T. Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl., **219** (1995), 139–155.
- [15] T. Furuta, *Simplified proof of an order preserving operator inequality*, Proc. Japan Acad. Ser. A Math. Sci., **74** (1998), 114.
- [16] T. Furuta, *Invitation to Linear Operators*, Taylor & Francis, London, 2001.
- [17] T. Furuta, *Monotonicity of order preserving operator functions*, Linear Algebra Appl., **428** (2008), 1072–1082.
- [18] T. Furuta, M. Hashimoto and M. Ito, *Equivalence relation between generalized Furuta inequality and related operator functions*, Sci. Math., **1** (1998), 257–259.
- [19] T. Furuta and D. Wang, *A decreasing operator function associated with the Furuta inequality*, Proc. Amer. Math. Soc., **126** (1998), 2427–2432.
- [20] T. Furuta, T. Yamazaki and M. Yanagida, *Order preserving operator function via Furuta inequality “ $A \geq B \geq 0$  ensures  $(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$  for  $p \geq 1$  and  $r \geq 0$ ”*, Proc. 96-IWOTA, 175–184.
- [21] M. Ito and E. Kamei, *A complement to monotonicity of generalized Furuta-type operator functions*, Linear Algebra Appl., **430** (2009), 544–546.
- [22] E. Kamei, *A satellite to Furuta’s inequality*, Math. Japon., **33** (1988), 883–886.



- [23] E. Kamei, *Extension of Furuta inequality via generalized Ando-Hiai theorem (Japanese)*, Sūrikaiseikikenkyūsho Kōkyūroku, Research Institute for Mathematical Sciences, **1535** (2007), 109–111.
- [24] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., **246** (1980), 205–224.
- [25] K. Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141–146.
- [26] K. Tanahashi, *The best possibility of the grand Furuta inequality*, Proc. Amer. Math. Soc., **128** (2000), 511–519.
- [27] M. Uchiyama, *Some exponential operator inequalities*, Math. Inequal. Appl., **2** (1999), 469–471.
- [28] T. Yamazaki, *Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality*, Math. Inequal. Appl., **2** (1999), 473–477.
- [29] J. Yuan and Z. Gao, *Classified construction of generalized Furuta type operator functions*, Math. Inequal. Appl., **11** (2008), 189–202.

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